Advanced Mathematics

- ***Differential equation*** is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders. Differential equations play a prominent role in engineering, physics, economics, and other disciplines.

We can write it in the following form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1)$$

Where F the real function depend on $(n+2)$ variables $(x, y, \frac{dy}{dx}, \dots, \frac{d^ny}{dx^n})$, n positive integer number.

- ***Classification of Differential equation***

We can classification in two types ***Ordinary and Partial Differential Equation*** one of the more obvious classifications is based on whether the unknown function depends on a single independent variable or on several independent variables. In the first case, only ordinary derivatives appear in the differential equation, and it is said to be an ***ordinary differential equation***. In the second case, the derivatives are partial derivatives, and the equation is called a ***partial differential equation***.

Examples:

a. $\frac{dy}{dx} - 2x = 0$

b. $\frac{d^2y}{dx^2} = ky$

c. $(\frac{\partial y}{\partial x})^2 = 4\left(\frac{\partial y}{\partial x}\right) - 2y^2 + 4x$

d. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 12z$

e. $(3x - 2y + 3)dx - (2x + 3y - 1)dy = 0$

Note: 1- The equation in a and e are ***ordinary differential equation*** that ***first order*** and ***first degree***.

2- The equation in b is an ***ordinary differential equation*** that ***second order*** and ***first degree.***

3- The equation in c is ***partial differential equation*** that ***first order*** and ***second degree.***

4- The equation in d is ***partial differential equation*** that ***first order*** and ***first degree.***

Order: The ***order*** of a differential equation is the order of the highest derivative that appear in equation.

Degree: The ***degree*** of a differential equation is the degree of the exponent highest derivative that appear in equation.

Example: Consider the following differential equation

$$\frac{d^2y}{dx^2} = k\left[1 - \left(\frac{\partial y}{\partial x}\right)^2\right]^{5/2}$$

by taking the square, we get

$$\left(\frac{d^2y}{dx^2}\right)^2 = k^2\left[1 - \left(\frac{\partial y}{\partial x}\right)^2\right]^5$$

So that, the degree of the differential equation that is given is the ***second degree.***

- ***Ordinary Differential Equation of First Order:*** We can write in general in the following form

$$\frac{dy}{dx} = f(x,y)$$

or

$$M(x,y)dx + N(x,y)dy = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Note: $\frac{dy}{dx} = y' = y_x$

Example:

a. $\frac{dy}{dx} - 2x = 0 \rightarrow \frac{dy}{dx} = 2x$

e. $(3x - 2y + 3)dx - (2x + 3y - 1)dy = 0$

$\frac{dy}{dx} = \frac{3x-2y+3}{2x+3y-1}$

- ***Methods of Solution of an Ordinary Differential Equation of First Order***

❖ ***Separation of variables:*** If we can write an ordinary differential equation of first order that given in eq(2) in the form

$$M(x)dx + N(y)dy = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Where M function of x only, N function of y only, we say that x and y in eq(3) are separation of variables.

Note: We can solution eq(3) by using integration.

Example: Find the solution of the following equation

$$(x-1)^2 y\,dx + x^2(y+1)dy = 0 \text{ , where } x \neq 0 \text{ and } y \neq 0$$

Solution: we can rewrite the above equation in the form

$$\frac{(x-1)^2}{x^2}dx + \frac{(y+1)}{y}dy = 0$$

$$\frac{x^2-2x+1}{x^2}dx + \frac{(y+1)}{y}dy = 0$$

$\left(1-\frac{2}{x}+\frac{1}{x^2}\right)dx + \left(1+\frac{1}{y}\right)dy = 0$ by using integration, we get

$$\int\left(1-\frac{2}{x}+\frac{1}{x^2}\right)dx + \int\left(1+\frac{1}{y}\right)dy = c$$

$$x - 2lnx - \frac{1}{x} + y + lny = c$$

$\frac{y}{x^2} = e^c e^{\frac{1}{x}-y-x} = ke^{\frac{1}{x}-y-x}$, where $k = e^c$

Example: Solve the initial value problem

$$(1+x^3)dy - x^2y dx = 0\ , \quad y(1) = 2$$

Solution: we can rewrite the above equation in the form

$\frac{x^2}{1+x^3}dx - \frac{1}{y}dy = 0$ by using integration, we get

$y = \frac{1}{A}(1+x^3)^{1/3}$, where $A = e^c$

By using the initial condition, we get

$$2 = \frac{1}{A}(1+(1)^3)^{1/3} \rightarrow A = 2^{-2/3}$$

Hence , the particular solution of the equation is

$$y = 2^{-2/3}(1+x^3)^{1/3}$$

Problem: Solve the initial value problem

$$\frac{dy}{dx} = \frac{y\,cosx}{1+2y^2}\ , \quad y(0) = 1$$

- ***Homogeneous First-Order Equation***

Definition: Homogeneous function $f(x,y)$ in x, y that degree n if and only if $f(\lambda x, \lambda y) = \lambda^n f(x,y)$, where $\lambda > 0$.

Example: Let $f(x,y) = 4x^3 - 3xy^2 + 5y^3$

$$\begin{aligned} f(\lambda x, \lambda y) &= 4(\lambda x)^3 - 3(\lambda x)(\lambda y)^2 + 5(\lambda y)^3 \\ &= \lambda^3(4x^3 - 3xy^2 + 5y^3) \\ &= \lambda^3 f(x,y) \text{ homogeneous function of third degree.} \end{aligned}$$

Example: Let a. $f(x,y) = \frac{xe^{y/x}+2y}{x^2}$

b. $f(x,y) = \frac{\sqrt{x^2+y^2}}{x} + \sin^2\frac{y}{x}$

Show that: 1-f in a is homogeneous and has -1 degree.

2- f in b is homogeneous and has 0 degree.

Definition: The following differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is called homogeneous differential equation of first order if and only if M, N homogeneous functions have the same degree.

Steps solution to homogeneous differential equation:

Let $M(x,y)dx + N(x,y)dy = 0$………………(*)

homogeneous differential equation of first order

1- Suppose that $y = Vx$, where V a new assistant variable.

2- $\frac{dy}{dx} = V + x\frac{dV}{dx}$.

3- Substitute (1) and (2) in equation (*) the equation (*) transform to a new equation in variables x and V, where the new equation can solve it by using separation of variables.

4- Substitute $\frac{y}{x}$ instead of V.

Example: Solve the following differential equation

$$(x^3 + y^3)dx - 3xy^2dy = 0 \text{ …………..(a)}$$

Solution: Since $M(x,y) = x^3 + y^3$ and $N(x,y) = -3xy^2$ are homogeneous functions has third degree

∴ eq(a) is homogeneous equation, and we can rewrite eq(a) in the following form

$$\frac{dy}{dx} = \frac{x^3+y^3}{3xy^2} \text{ …………(b)}$$

Let $y = Vx \rightarrow \frac{dy}{dx} = V + x\frac{dV}{dx}$ by substitute, we get

$$V + x\frac{dV}{dx} = \frac{x^3+(Vx)^3}{3x(Vx)^2} = \frac{1+V^3}{3V^2} \rightarrow x\frac{dV}{dx} = \frac{1+V^3}{3V^2} - V$$

$$\frac{3V^2}{1-2V^3}dV - \frac{dx}{x} = 0 \text{………..(c)}$$

by using integration, we get

$(1 - 2V^3)(x^2) = e^{-2c} = k$ the general solution to eq(c) since $y = Vx$, $\therefore \left(1 - 2\frac{y^3}{x^3}\right)(x^2) = k$ the general solution to eq(a).

Example: Solve the following differential equation

$$\theta\frac{dt}{d\theta} = t - \theta cos^2\frac{t}{\theta} \text{ ....(a)}$$

Solution: We can rewrite eq(a) in the following form

$$\frac{dt}{d\theta} = \frac{t}{\theta} - cos^2\frac{t}{\theta} = f(t,\theta)$$

Since $f(t,\theta)$ is homogeneous function has zero degree, so that the differential equation in (a) is also homogeneous

let $t = V\theta \rightarrow \frac{dt}{d\theta} = V + \theta\frac{dV}{d\theta} \rightarrow \frac{V\theta}{\theta} - cos^2\frac{V\theta}{\theta} = V + \theta\frac{dV}{d\theta} \rightarrow$

$\frac{1}{cos^2V}dV + \frac{1}{\theta}d\theta = 0$, by using integration, we get

$tanv + ln\theta = c \rightarrow ln\theta = c - tanv \rightarrow \theta = e^{c-tanv} = Ae^{-tanv} \rightarrow$

$\theta = Ae^{-tan\frac{t}{\theta}}$, where $A = e^c$ and $V = \frac{t}{\theta}$.

Problem: 1. Show that the given equation is homogeneous.

2. Solve the differential equation.

a. $\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$

b. $\frac{dy}{dx} = \frac{4y-3x}{2x-y}$

❖ ***Exact Differential Equation***

Definition: said that the differential equation of first order

$$M(x,y)dx + N(x,y)dy = 0$$

is exact in the region D if and only if there ***exist*** function denoted by $u = u(x,y)$ such that for all point in this region satisfy $\frac{\partial u}{\partial x} = M$, $\frac{\partial u}{\partial y} = N$.

Theorem: Let the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

exact in the region D then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Example: Consider the following differential equation

$$(2xy + 3y^3)dx + (x^2 + 9xy^2)dy = 0$$

$M(x,y) = 2xy + 3y^3 \rightarrow \frac{\partial M}{\partial y} = 2x + 9y^2$

$N(x,y) = x^2 + 9xy^2 \rightarrow \frac{\partial N}{\partial x} = 2x + 9y^2$

Since, $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ so that, differential equation is exact.

Steps solution to exact differential equation:

Suppose that $M(x,y)dx+N(x,y)dy=0$ …(a) an exact differential equation of first order and let $\mathrm{w(x,y)=c}$ represent solution to exact eq(a) by using the chain rule, we get $\partial w=\frac{\partial w}{\partial x}dx+\frac{\partial w}{\partial y}dy=0$….(b) comparing (a) and (b) we get

$\frac{\partial w}{\partial x}=M(x,y)\rightarrow\partial w=M(x,y)\partial x$ …..(c1)

$\frac{\partial w}{\partial y}=N(x,y)\rightarrow\partial w=N(x,y)\partial y$ ……(c2)

by using integration to eq(c1), we get

$$\int dw=\int M(x,y)dx+\varphi(y)$$

$$w(x,y)=\int M(x,y)dx+\varphi(y)\text{……(d)}$$

derivative eq(d) with respect to y, we get

$\frac{\partial w}{\partial y}=\frac{\partial}{\partial y}\int M(x,y)dx+\varphi'(y)=N(x,y)$ …..(e)

from equations (d) and (e) we find $\varphi(y)$ therefore, substitute the value of $\varphi(y)$ in eq(d) to get the solution $\mathrm{w(x,y)=c}$ to differential equation (a).

Example: Solve the following differential equation

$$(x^2-y)dx+(y^2-x)dy=0$$

Solution: Since $M(x,y)=x^2-y\rightarrow\frac{\partial M}{\partial y}=-1$ and

$$N(x,y)=y^2-x\rightarrow\frac{\partial N}{\partial x}=-1$$

∴ the equation is an exact. Hence let $\mathrm{w(x,y)=c}$ represent solution to the given equation

$w(x,y)=\int M(x,y)dx+\varphi(y)=\int(x^2-y)dx+\varphi(y)$

$w(x,y)=\frac{x^3}{3}-xy+\varphi(y)$……..(*)

derivative eq(*) with respect to y, we get

$$\frac{\partial w(x,y)}{\partial y}=-x+\varphi'(y)=y^2-x=N(x,y)\rightarrow\varphi'(y)=y^2\rightarrow\varphi(y)=\frac{y^3}{3}$$

substitute the value of $\varphi(y)$ in eq(*) we get $w(x,y)=\frac{x^3}{3}-xy+\frac{y^3}{3}$

therefore, the general solution to the differential equation is $\frac{x^3}{3}-xy+\frac{y^3}{3}=c$.

Problem: Solve the following differential equation

$$(ycosx + 2xe^y)dx + (sinx + x^2e^y - 1)dy = 0$$

- ***Differential equation of the first order and removable turned into an exact equation***

Definition: Let $M(x,y)dx + N(x,y)dy = 0$ …(a) differential equation of the first order. Said that the eq(a) not exact if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Question: Is it can transform not exact equation to an exact?

Answer: Yes.

Definition: Let $M(x,y)dx + N(x,y)dy = 0$ differential equation of the first order is not exact. If there exist function $\mu = \mu(x,y)$ such that $\mu(x,y)[M(x,y)dx + N(x,y)dy] = 0$ ……….(b) called an exact equation and $\mu(x,y)$ called integral factor.

How to find an integral factor: By using definition of an exact equation we will get $\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x}$ hence, $\mu\frac{\partial M}{\partial y} + M\frac{\partial \mu}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x} \to$

$\frac{1}{\mu}(N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y}) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ …....(c)

There exist two cases: The first one if the integral factor μ function of x only

$\therefore \frac{\partial \mu}{\partial x} = \frac{d\mu}{dx}$, $\frac{\partial \mu}{\partial y} = 0$, by substitute in eq(c), we get

$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N(x,y)} = f(x)$ ……(d) we can write eq(d) in the following form

$\frac{d\mu}{\mu} = f(x)dx$ that it is separation of variables equation

$\therefore \mu(x) = e^{\int f(x)dx}$ …..(e)

The last one if the integral factor μ function of y only by using the same way

$\frac{\partial \mu}{\partial x} = 0, \frac{\partial \mu}{\partial y} = \frac{d\mu}{dy}$

$\therefore \frac{1}{\mu}\left(-M\frac{d\mu}{dy}\right) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \to \frac{1}{\mu}\frac{d\mu}{dy} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M(x,y)} = g(y)$ ……(f)

$\therefore$ the integral factor become $\mu(y) = e^{\int g(y)dy}$ ………..(g)

Problem: Solve the following differential equation

$$(x^2 + y^2 + x)dx + (xy)dy = 0$$

Solution: Since $\frac{\partial M}{\partial y} = 2y \neq \frac{\partial N}{\partial x} = y$, hence the equation is not exact.

$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N(x,y)} = \frac{2y-y}{xy} = \frac{1}{x} = f(x)$, thus the given equation removable turned into an exact equation and the integral factor is given by $\mu(x) = e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} \rightarrow$
$\mu(x) = e^{lnx} = x$
$\therefore x(x^2 + y^2 + x)dx + x(xy)dy = 0 \rightarrow$
$(x^3 + xy^2 + x^2)dx + (x^2y)dy = 0$ ……(*) where, $\frac{\partial \mu M}{\partial y} = 2xy = \frac{\partial \mu N}{\partial x}$
hence, eq(*) an exact equation. The solution of eq(*) is given by

$$w(x,y) = \int \mu M(x,y)dx + \varphi(y) = (x^3 + xy^2 + x^2)dx + \varphi(y)$$

$$= \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} + \varphi(y)$$

$$\frac{\partial w}{\partial y} = x^2y + \varphi'(y) = \mu N(x,y) = x^2y \rightarrow \varphi'(y) = 0 \rightarrow \varphi(y) = k$$

$\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} + k = c$ the general solution to the given differential equation.

Problem:

1-Solve the following differential equation

$$(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$$

2-Find an integrating factor for the equation

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

and then solve the equation.

❖ ***Linear Differential Equation of First Order***

Definition: The following differential equation $y' = \frac{dy}{dx} = f(x,y)$ is called linear equation if the function $f(x,y)$ is linear function, and we can write it in the following form $y' + P(x)y = q(x)$ …..(a) , where $P(x)$, $q(x)$ continuous function on the interval I.

Method of solution of linear equation of first order

We can rewrite the eq(a) in the following form $\big(P(x)y - q(x)\big)dx + dy = 0$
since $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N(x,y)} = \frac{P(x)}{1} = P(x)$ hence, the integration factor is given by
$\mu(x) = e^{\int P(x)dx}$ therefore, when multiply the integral factor by eq(a) we get
$e^{\int P(x)dx}\, \frac{dy}{dx} + e^{\int P(x)dx}\, P(x)y = e^{\int P(x)dx}\, q(x)$

$\frac{d}{dx}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}q(x)$, so that the general solution of eq(a) is given by $ye^{\int P(x)dx} = \int e^{\int P(x)dx}q(x)\,dx + c$

$ye^{\int P(x)dx} = e^{-\int P(x)dx}[\int e^{\int P(x)dx}q(x)\,dx + c]$…..(b)

$$y = [\mu(x)]^{-1}\int \mu(x)q(x)\,dx + c$$

Example: solve the following equation

$$\frac{dy}{dx} + 2xy = 4x$$

Solution: Since $P(x) = 2x$ and $q(x) = 4x$ are continuous on the interval I that is equal to R

$\therefore \mu(x) = e^{\int P(x)dx} = e^{\int 2xdx} = e^{x^2} \rightarrow ye^{x^2} = \int e^{x^2}(4x)dx + c = 2e^{x^2} + c$

$y = 2 + ce^{-x^2}$

Problem:

1- Solve the following equation

$$\frac{dy}{dx} + \frac{y}{x} = 3x$$

Hint: $P(x) = \frac{1}{x}$ continuous for values $x \neq 0 \in R$, $q(x) = 3x$ continuous for all values $x \in R$.

2- Solve the initial value problem

$y' + 2y =$g(t), $y(0) = 0$,

Where

$$g(t)=\begin{cases}1, & 0 \leq t \leq 1\\ 0, & t > 1\end{cases}$$

❖ *Bernoulli Equations*

Sometimes it is possible to solve a nonlinear equation by making a change of the dependent variable that converts into a linear equation. The most important such equation has the form

$\frac{dy}{dx} + P(x)y = y^n q(x)$ ……(a)

and is called a Bernoulli Equation.

first dividing both sides of eq(a) by y^n, where n positive integer number, then let $V = \frac{1}{y^{n-1}} = y^{-(n-1)}$ ……(b)

$\therefore \frac{dV}{dx} = -(n-1)y^{-n}\frac{dy}{dx} \rightarrow y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dV}{dx}$ ……(c)

from eq(a) we get $y^{-n}\frac{dy}{dx} + P(x)y^{-(n-1)} = q(x)$ comparison equations b, c, and last equation we get

$\frac{1}{1-n}\frac{dV}{dx} + P(x)V = q(x)$ this is linear equation in variables $V$and x.

Example: Solve the following equation

$$\frac{dy}{dx} + \frac{1}{3}y = \frac{1}{3}(1-2x)y^4$$

Solution: divide given equation on $y^4 \neq 0$ we get

$\frac{\mathbf{1}}{y^4}\frac{dy}{dx} + \frac{1}{3}\frac{1}{y^3} = \frac{1}{3}(1-2x)$…….(*)

Let $V = \frac{1}{y^3} \rightarrow \frac{dV}{dx} = -3\frac{1}{y^4}\frac{dy}{dx}$ substitute the value of V and its derivative in eq(*) we get

$-\frac{1}{3}\frac{dV}{dx} + \frac{1}{3}V = \frac{1}{3}(1-2x) \rightarrow \frac{dV}{dx} - V = 2x - 1$ this is linear equation

$P(x) = -1,\ q(x) = 2x - 1 \rightarrow \mu(x) = e^{\int -dx} = e^{-x}$

$V\mu(x) = \int \mu(x)\, q(x)dx + c \rightarrow Ve^{-x} = \int e^{-x}(2x-1)dx + c$

$Ve^{-x} = \int 2xe^{-x}dx - \int e^{-x}dx + c$

$Ve^{-x} = 2[-xe^{-x} - \int -e^{-x}\, dx] - \int e^{-x}dx + c$

$Ve^{-x} = 2[-xe^{-x} - e^{-x}] + e^{-x} + c$

$Ve^{-x} = -2xe^{-x} - 2e^{-x} + e^{-x} + c$

$Ve^{-x} = -2xe^{-x} - e^{-x} + c \rightarrow V = -2x - 1 + ce^{x}$

$$\frac{1}{y^3} = -2x - 1 + ce^x$$

Problem: Solve the following equation

$$x\frac{dy}{dx} + y = xy^3$$

Initial value problems (IVP) and Boundary value problem (BVP)

Example: Find the solution $\varphi(x) = y$ to the following equation $\frac{dy}{dx} = 3x^2$ and satisfy the condition $\varphi(2) = 3$

Since the given equation is separation of variable so that the solution it is given by $y = \varphi(x) = x^3 + c$ this is general solution. When substitute the initial condition $x = 2 \rightarrow y = 3, 3 = 8 + c \rightarrow c = -5$ therefore, the particular solution is given by $\varphi(x) = x^3 - 5$.

Obviously the solution give us curve called ***integral curve*** that pass in the point $(2,3) = (x_0, y_0)$.

When studying the differential equation of first order or of highest order as in the previous example differential equation that have one condition or more and ask us find the solution ***integral curve*** that satisfy the given equation and also the given condition.

Note: If the conditions that it is given with the differential equation bounded by one value of the independent variable the problems called initial value problem or if the differential equation bounded by more one value of condition of the independent variable then the problems called boundary value problem.

Examples:

1- $\frac{\mathrm{dy}}{\mathrm{dx}} = -\frac{\mathrm{x}}{\mathrm{y}}$, $y(3) = 4$ initial value problem.

2- $\frac{d^2y}{dx^2} \neq \frac{\mathrm{dy}}{\mathrm{dx}} = 0$, such that $y(1) = 4, y(1) = 3$ initial value problem.

3- $\frac{d^2y}{dx^2} + \frac{\mathrm{dy}}{\mathrm{dx}} = 0$, such that $y\left(\frac{\pi}{2}\right) = 5, y(0) = 1$ boundary value problem.

So that the general form of the initial value problem of the first order is given by

$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$……(*)

Definition: Let φ function defined on the interval I and contain point x_0 we say that φ solution to the initial value problem (*) if φ solution to equation $\frac{dy}{dx} = f(x, y)$ and satisfy the initial condition $\varphi(x_0) = y_0$.

Example: Solve the following equation

$$y' = 2x, \quad y(2) = 3$$

- ***Linear Differential Equation of nth Order***

  Definition: We can define the linear differential equation of nth order in dependent variable y and independent variable x in the form

  $$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = h(x) \ldots\ldots(1)$$

  Where $a_0, a_1, \ldots, a_n, h$ continuous functions in the interval I and $a_0(x) \neq 0$ $\forall x \in I$, when $h(x) = 0 \ \forall x \in I$ the equation

  $$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \ldots\ldots.(2)$$

  called homogeneous linear differential equation of nth order, otherwise it is called non homogeneous linear differential equation.

  Example: $3y'' - 2xy' + x^2y = xe^x$ non homogeneous linear equation of second order.

  $3y'' - 2xy' + x^2y = 0$ homogeneous linear equation of second order.

Note: 1- The continuous functions $a_i(x)\ \forall\ i = 0,1,2,\dots,n$ called derivative coefficient $\frac{d^{n-i}y}{dx^{n-i}}$ $(i = 0,1,\dots,n)$ and if Ln written influential defined in the form $Ln = a_0(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{d}{dx} + a_n(x)$ therefore, the eq(1) written as $Lny = h(x)$.

2- if φ function defined on the interval I and belong to $C^n(I)$ and satisfy eq(1) then φ called solution of this equation.

❖ ***Linear Differential Equation of nth Order with Constant Coefficient***

The general form of this type of equation is given by

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = h(x)\ \dots\dots(1)$$

Where $a_0,\ a_1,\dots,a_n$ are constant and $a_0 \neq 0$, and h continuous function defined on the interval I.

By using written influential the eq(1) written as

$$Lny = h(x)\ \dots\dots(2)$$

eq(2) called non homogenous linear differential equation when $h(x) \neq 0$ otherwise, eq(2) called homogenous linear differential equation, that is $Lny = 0\dots\dots(3)$.

Solution of homogenous linear differential equation

The eq(3) have n roots that linear independent on the interval I. Let $y = e^{\lambda x}$ solution to the eq(3), where λ constant number therefore,

$Ln(e^{\lambda x}) = 0$. Since $Ln = a_0\frac{d^n}{dx^n} + a_1\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1}\frac{d}{dx} + a_n$

$\therefore \frac{\partial}{\partial x}(e^{\lambda x}) = \lambda e^{\lambda x}$, $\frac{\partial^2}{\partial x^2}(e^{\lambda x}) = \lambda^2 e^{\lambda x}$, …., $\frac{\partial^n}{\partial x^n}(e^{\lambda x}) = \lambda^n e^{\lambda x}$, therefore the homogenous equation becomes

$e^{\lambda x}(a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n) = 0$, since $e^{\lambda x} \neq 0\ \forall x$

$P_n(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0\dots\dots(4)$

eq(4) is called the characteristic equation of the homogenous linear differential eq(3), where eq(4) be necessary and sufficient condition in order to be $e^{\lambda x}$ solution to eq(3). Eq(4) represent polynomial of degree n and have n roots which are either to be distinct (real, complex) or repeated (real, complex).

A) If the roots equation $P_n(\lambda) = 0$ have distinct values then eq(3) have n linear independent solutions that it is

$$y_i(x) = e^{\lambda_i x} \qquad i = 1,2,3,\ldots$$

and $\quad y = \sum_{i=1}^{n} c_i\, e^{\lambda_i x}$ also represent general solution to eq(3).

Example: Solve the following differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 10y = 0$$

Solution: Since the characteristic equation of this equation is given by

$$\lambda^2 - 3\,\lambda - 10 = 0 \rightarrow (\lambda - 5)(\lambda + 2) = 0$$

Then the roots of $P_2(\lambda) = 0$ are given by $\lambda_1 = 5,\quad \lambda_2 = -2$ and the solutions of the given differential equation written in the form

$$y_1(x) = e^{5\,x},\; y_2(x) = e^{-2\,x}$$

thus the general solution of linear differential equation is

$$y(x) = c_1 e^{5\,x} + c_2 e^{-2\,x}$$

Example: Solve the following equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$$

Solution: Since the characteristic equation of this equation is given by

$$\lambda^2 + 4\,\lambda + 5 = 0$$

Then the roots are given by $\lambda_1 = -2 + i, \quad \lambda_2 = -2 - i$

$\therefore$ the solutions are $y_1(x) = e^{(-2+i)\,x},\;\; y_2(x) = e^{(-2-i)\,x}$ thus the general solution of linear differential equation is

$$y = c_1 e^{(-2+i)\,x} + c_2 e^{(-2-i)\,x} \rightarrow y = e^{-2}(c_1 e^{ix} + c_2 e^{-ix})$$

Note: $e^{ix} = cosx + isinx,\;\; e^{-ix} = cosx - isinx$

Example: Solve the following differential equation

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

Solution: Since the characteristic equation of this equation is given by

$P_3(\lambda) = \lambda^3 - \lambda^2 - 4\,\lambda + 4 = 0 \rightarrow \lambda^2(\lambda - 1) - 4(\lambda - 1) = 0 \rightarrow$
$(\lambda - 1)(\lambda^2 - 4) = 0 \rightarrow (\lambda - 1)(\lambda - 2)(\lambda + 2) = 0$ thus the roots of $P_3(\lambda) = 0$ are $\lambda_1 = 1,\;\; \lambda_2 = -2,\;\; \lambda_3 = 2$ and the solutions to the given equation are $y_1(x) = e^{\,x},\;\; y_2(x) = e^{-2x},\;\; y_3(x) = e^{2x}$ therefore, the general solution is given by $y = c_1 e^{\,x} + c_2 e^{-2x} + c_3 e^{2x}$

Example: Solve the following differential equation

$$\frac{d^3y}{dx^3} + (1+3i)\frac{d^2y}{dx^2} + (3i-2)\frac{dy}{dx} - 6i = 0$$

Solution: Since the characteristic equation of this equation is given by

$P_3(\lambda) = \lambda^3 + (1+3i)\lambda^2 + (3i-2)\,\lambda - 6i = 0$ polynomial of third degree,

note that $\lambda = 1$ satisfy the equation such that

$1 + (1+3i) + (3i-2) - 6i = 0$ therefore, $\lambda = 1$ one of the root the equation $P_3(\lambda)$ to find the others roots using long division, so that we get

$P_3(\lambda) = (\lambda - 1)(\lambda^2 + (2+3i)\lambda + 6i = 0$

$= (\lambda - 1)(\lambda + 3i)(\lambda + 2) = 0$ and the roots are $\lambda_1 = 1,\ \lambda_2 = -2,$ $\lambda_3 = -3i$ thus the solutions of the given equation are $y_1(x) = e^{\,x}$, $y_2(x) = e^{-2\,x}$, $y_3(x) = e^{-3ix}$ therefore the general solution is given by

$y(x) = c_1 e^{\,x} + c_2 e^{-2\,x} + c_3 e^{-3ix}$

B) If the roots equation $P_n(\lambda) = 0$ have repeated values. Consider the following equation $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$…….(*) or written by $Lny = 0$ such that $P_2(\lambda) = \lambda^2 + a\lambda + b = 0$ represent the characteristic equation to the given equation (*) which can find its roots in the form

$\lambda_1, \lambda_2 = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$, $\therefore\ \ a = -(\lambda_1 + \lambda_2)$

i) When $a^2 - 4b = 0$ the roots λ_1, λ_2 are real and equal (i.e $\lambda_1 = \lambda_2$) thus $y_1 = e^{\,\lambda_1 x}$ represent solution to eq(*) to find the second root we can use

$$y_2(x) = y_1(x)\int \frac{\exp(-\int \frac{a_1(x)}{a_0(x)}dx)}{{y_1}^2}dx = e^{\,\lambda_1 x}\int \frac{\exp(-adx)}{e^{2\lambda_1 x}}dx$$

$$= e^{\lambda_1 x} \int e^{-2\lambda_1 x} e^{-ax} dx = e^{\lambda_1 x} \int e^{-2\lambda_1 x} e^{2\lambda_1 x} dx = e^{\lambda_1 x} \int dx$$

$= xe^{\lambda_1 x}$ therefore, the general solution to eq(*) is given by

$$y = c_1 e^{\lambda_1 x} + c_2 x\, e^{\lambda_1 x} = (c_1 + c_2 x)e^{\lambda_1 x}$$

ii) When $a^2 - 4b < 0$ then the roots will be complex number

i.e $\lambda_1, \lambda_2 = \beta + i\mu$ where $\beta = \frac{-a}{2}$, $\mu = \frac{\sqrt{a^2-4b}}{2}$ thus the general solution

is $y = c_1 e^{(\beta+i\mu)x} + c_2 e^{(\beta-i\mu)x} = c_1 e^{\beta x} e^{i\mu x} + c_2 e^{\beta x} e^{-i\mu x} \rightarrow$

$$y = e^{\beta x}\left(c_1 e^{i\mu x} + c_2 e^{-i\mu x}\right) = e^{\beta x}((c_1 + c_2)cos\mu x + i(c_1 - c_2)sin\mu x)$$

$$y = e^{\beta x}(k_1 cos\mu x + k_2 sin\mu x).$$

In general form if the equation $L3y = 0$ and have three repeated roots the general solution is given by $y = (c_1 + xc_2 + x^2 c_3)\, e^{\lambda_1 x}$.

Theorem: Let $Lny = 0$ homogenous linear differential equation of *nth* order and that type constant coefficient and if the characteristic equation have λ repeated k times then the general solution that similar to repeated roots k times given in the form $(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1})\, e^{\lambda x}$ and if the others roots to the characteristic equation distinct i.e $(\lambda_{k+1}, \lambda_{k+1}, \ldots, \lambda_n)$ then the general solution to the equation $Lny = 0$ is given by the following form

$$y = (c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1})e^{\lambda x} + c_{k+1} e^{\lambda_{k+1} x} + \cdots + c_n e^{\lambda_n x}$$

Example: Solve the following differential equation

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 8y = 0$$

Solution: the characteristic equation to the given equation is

$P_3(\lambda) = \lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$

$= \lambda^2(\lambda - 2) - 4(\lambda - 2) = 0 \to (\lambda^2 - 4)(\lambda - 2) = 0 \to$

$= (\lambda - 2)(\lambda + 2)(\lambda - 2) = 0 \to (\lambda - 2)^2(\lambda + 2) = 0$ thus the roots are $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = -2$ therefore, the general solution is given by $y = (c_1 + c_2 x)e^{2x} + c_3 e^{-2x}$

Example: Solve the following initial value problem

$$y'' + 4y' + 4y = 0, \quad y'(0) = -5, \ y(0) = 4$$

Solution: the characteristic equation to the given equation is

$\lambda^2 + 4\lambda + 4 = 0 \to (\lambda + 2)^2 = 0 \to \lambda_1 = \lambda_2 = -2$
$\therefore$ the general solution is $y = (c_1 + c_2 x)e^{-2x}$
since $y' = -2c_1 e^{-2x} - 2c_2 x\, e^{-2x} + c_2 e^{-2x}$ substitute the initial conditions we get $y(0) = c_1 = 4$, $y'(0) = -2c_1 + c_2 = -5 \to c_2 = 3$
therefore, the unique solution to the initial value problem is given by
$y = 4e^{-2x} + 3x\, e^{-2x} \to y = e^{-2x}(4 + 3x)$
Problem: Solve the following equation

$$y'' - 2y' + 10y = 0$$

- ***Non Homogenous Linear Differential Equation with Constant Coefficient***

The general form of this type of equation is written as
$a_0 \frac{d^n}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_n y = h(x)$ or $Lny = h(x)$……(1)
Where, $a_0, a_1, \dots, a_n$ are constant and h continuous function defined on the interval I .
The general solution of equation (1) consist of from the complement function which is general solution of the homogeneous equation (1) (i.e $Lny = 0$) and the particular solution to the non homogeneous equation (1) such that
$y(x) = y_c(x) + y_p(x)$ where, $y_c(x)$ general solution of the homogeneous equation (1) and $y_p(x)$ particular solution to the non homogeneous equation (1) .
To find the particular solution to the non homogeneous equation there is two cases one of them called general and the others called special

Case1: Method of variation of constant (the general method)

This method biased on the idea change the arbitrary constant c_i in the general solution $\sum_{i=1}^{n} c_i\, g_i$ of the homogeneous equation $Lny = 0$ to the functions v_i such that $\sum_{i=1}^{n} v_i\, g_i$ in order to be particular solution to non homogeneous equation (1).

Definition: Let $g_1,\ g_2, \dots,\ g_n$ functions in $C^{n-1}(I)$ and let

$$w(g_1,\ g_2, \dots,\ g_n)(x) = \begin{vmatrix} g_1(x) & g_2(x) & \dots & g_n(x) \\ g_1'(x) & g_2'(x) & \dots & g_n'(x) \\ \vdots & \vdots & & \vdots \\ g_1^{n-1}(x) & g_2^{n-1}(x) & & g_n^{n-1}(x) \end{vmatrix}$$

$\forall x \in I$ we said that $w(g_1,\ g_2, \dots,\ g_n)(x)$ Wronskian determent of this functions.

Example: $\forall x \in (-\infty, \infty)$ then $w(x, sinx) = \begin{vmatrix} x & sinx \\ 1 & cosx \end{vmatrix} = xcosx - sinx$ called Wronskian of the functions x and $sinx$.

Example: Wronskian of the functions $cosx$ and $sinx$ is

$$w(cosx, sinx) = \begin{vmatrix} cosx & sinx \\ -sinx & cosx \end{vmatrix} = cos^2x + sin^2x = 1$$

Example: Wronskian of the functions $e^x$and e^{-x} is

$$w(e^x, e^{-x}) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

Theorem: Let $g_1,\ g_2$ linear independent solutions of equation $L_2y = 0$ on the interval *I*. Then the particular solution y_p of the non homogeneous equation $L_2y = h(x)$ is given by the form $y_p(x) = \int_{x_0}^{x} \frac{[g_1(t)\ g_2(x) - g_1(x)\ g_2(t)]}{a_0\, w(g_1, g_2)(t)}\, h(t)dt$ where, x_0 known point in an interval *I*.

Proof: Let $V_1,\ V_2$ functions belong to $C'(I)$ provided that

$y_p(x) = V_1(x)\, g_1(x) + V_2\, g_2(x)$……..(a) and

$V'_1(x)\, g_1(x) + V'_2(x)\, g_2(x) = 0$……(b) since y_p represent solution to non homogeneous equation $L_2y = h(x)$

$\therefore\ a_0\, y_p''(x) + a_1\, y_p'(x) + a_2 y_p(x) = h(x) \quad \forall\, x \in I$ …….(c) since

$y_p'(x) = \boldsymbol{V'_1(x)\, g_1(x)} + V_1(x)\, g'_1(x) + \boldsymbol{V'_2(x) g_2(x)} + V_2(x)\, g'_2(x)$

$\qquad = V_1(x)\, g'_1(x) + V_2(x)\, g'_2(x)$

$y_p''(x) = V'_1(x)\, g'_1(x) + V_1(x)\, g''_1(x) + V'_2(x) g'_2(x) + V_2(x)\, g''_2(x)$

substitute the values of $y_p(x), y_p'(x)$ and $y_p''(x)$ in eq(c) we get

$$V_1(x)[\boldsymbol{a_0 g_1'' + a_1 g_1' + a_2\, g_1}] + V_2(x)[\boldsymbol{a_0 g_2'' + a_1 g_2' + a_2\, g_2}] + a_0\left[V'_1(x)\, g'_1(x) + V'_2(x) g'_2(x)\right] = h(x) \rightarrow$$

$V'_1(x)\, g'_1(x) + V'_2(x) g'_2(x) = \frac{h(x)}{a_0}$……..(d) by using solution of linear system in (b) and (d) by Grammar method we get,

$$V'_1(x) = \frac{\begin{vmatrix} 0 & g_2(x) \\ \frac{h(x)}{a_0} & g'_2(x) \end{vmatrix}}{\begin{vmatrix} g_1(x) & g_2(x) \\ g'_1(x) & g'_2(x) \end{vmatrix}} = \frac{-g_2(x)h(x)}{a_0\, w(g_1,\, g_2)(x)} \rightarrow V_1(x) = \int_{x_0}^{x} \frac{-g_2(t)h(t)}{a_0\, w(g_1,\, g_2)(t)}\, dt$$

$$V'_2(x) = \frac{\begin{vmatrix} g_1(x) & 0 \\ g'_1(x) & \frac{h(x)}{a_0} \end{vmatrix}}{\begin{vmatrix} g_1(x) & g_2(x) \\ g'_1(x) & g'_2(x) \end{vmatrix}} = \frac{g_1(x)h(x)}{a_0\, w(g_1, g_2)(x)} \rightarrow V_2(x) = \int_{x_0}^{x} \frac{g_1(t)h(t)}{a_0\, w(g_1, g_2)(t)}\, dt$$

by substitute the values $V_1(x)$, $V_2(x)$ in (a) we get

$$y_p(x) = \int_{x_0}^{x} \frac{-g_2(t)h(t)}{a_0\, w\left(g_1,\, g_2\right)(t)}\, dt\;\, g_1(x) + \int_{x_0}^{x} \frac{g_1(t)h(t)}{a_0\, w(g_1,\, g_2)(t)}\, dt\;\; g_2(x)$$

$$= \int_{x_0}^{x} \frac{[g_1(t)\, g_2(x) - g_1(x) g_2(t)]}{a_0\, w(g_1, g_2)(t)}\; h(t)\, dt$$

Theorem: If $g_1, g_2, \dots, g_n$ linear independent solutions of the homogeneous equation $Lny = 0$ on the interval I then the particular solution of the non homogeneous equation $Lny = h(x)$ given by

$$y_p(x) = \sum_{k=1}^{n} g_k\,(x) \int_{x_0}^{x} \frac{w_k(t)h(t)}{a_0\, w(g_1, g_2, \dots, g_n)(t)}\, dt \quad …..(*)$$

Where, w_k is the hub that we get it from wronskian $w(g_1, g_2, \dots, g_n)$ by interchange the column $(g_k, g_k', \dots, g_k^{n-1})^t$ by column $(0, 0, \dots, 1)^t$.

Example: Find the general solution of the following equation

$$y'' - 3y' - 10y = e^x \qquad -\infty < x < \infty$$

Solution: The characteristic equation to homogeneous equation

$y'' - 3y' - 10y = 0$ is $\lambda^2 - 3\lambda - 10 = 0 \to (\lambda - 5)(\lambda + 2) = 0$ the roots are $\lambda_1 = 5, \quad \lambda_2 = -2$ thus, the solution of the homogeneous equation
$y'' - 3y' - 10y = 0$ is $g_1(x) = e^{5x}, \quad g_2(x) = e^{-2x}$

$$w(g_1,\ g_2) = w(e^{5x},\ e^{-2x}) = \begin{vmatrix} e^{5x} & e^{-2x} \\ 5e^{5x} & -2e^{-2x} \end{vmatrix} = -2e^{3x} - 5e^{3x} = -7e^{3x}$$

Using eq(*) in the previous theorem to find particular solution to non homogeneous equation, therefore

$$y_p(x) = \sum_{k=1}^{n} g_k(x) \int_{x_0}^{x} \frac{w_k(t)h(t)}{a_0\, w(g_1, g_2, \dots, g_n)(t)} dt$$

$$= \int_{x_0}^{x} \frac{g_1(t)\, g_2(x) - g_1(x)\, g_2(t)}{a_0\, w(g_1, g_2)(t)} h(t)dt$$

$$= \int_{0}^{x} \frac{[e^{5t}e^{-2x} - e^{5x}e^{-2t}]}{-7e^{3t}} e^t dt = -\frac{1}{12}e^x + \frac{e^{-2x}}{21} + \frac{e^{5x}}{28}$$

thus, the general solution to the non homogeneous equation $Lny = h(x)$ is

$$y(x) = y_c(x) + y_p(x) = c_1e^{5x} + c_2e^{-2x} - \frac{1}{12}e^x + \frac{e^{-2x}}{21} + \frac{e^{5x}}{28}$$

$$y(x) = \left(c_1 + \frac{1}{28}\right)e^{5x} + \left(c_2 + \frac{1}{21}\right)e^{-2x} - \frac{1}{12}e^x$$

Example: Find the particular solution of the following differential equation

$$y'' + y = tanx \quad \text{where,} \quad \frac{\pi}{2} < x < \frac{\pi}{2}$$

Solution: The characteristic equation of the homogeneous equation $y'' + y = 0$ is given by the following form $P_2(\lambda) = (\lambda^2 + 1) = 0$ and the roots are $y = \mp i$ therefore, the general solution of the homogeneous equation is

$y_c(x) = c_1 sinx + c_2 cosx,$ since $w(g_1,\ g_2)(x) = \begin{vmatrix} sinx & cosx \\ cosx & -sinx \end{vmatrix} = -1$

substitute the values of $w(g_1,\ g_2) = -1$, $g_1(x) = sinx$, $g_2(x) = cosx$, $h(x) = tanx$ in equation(*) we get the particular of the given non homogeneous equation

$$y_p(x) = \int_{x_0}^{x} \frac{(sint\, cosx - sinx\, cost)}{-1}\, tant\, dt = -cosx \int_0^x sint \left(\frac{sint}{cost}\right) dt + sinx \int_0^x cost \left(\frac{sint}{cost}\right) dt$$

$$= -cosx \int_0^x \left(\frac{sin^2t}{cost}\right) dt + sinx \int_0^x sint\, dt = -cosx \int_0^x \left(\frac{1 - cos^2t}{cost}\right) dt + sinx \int_0^x sint\, dt$$

$$= -cosx \int_0^x (sect - cosx)\, dt + sinx[-cost] \Big|_0^x$$

$$= -cosx[ln|sect - tant| - sint]\Big|_0^x + sinx(1 - cosx)$$
$$= -cosx[(ln|secx - tanx| - sinx) - (ln|1 - 0| - 0)] + sinx - sinx\ cosx$$
$$= -cosx\ ln|secx - tanx| + cosx\ sinx + sinx - sinx\ cosx$$
$$y_p(x) = sinx - cosx\ ln|secx - tanx|$$

Problem:

Find the general solution of the following equation

$$\frac{d^2y}{dx^2} + 4y = sec2x$$

Case2: Method of Undetermined Coefficients

The method of undetermined coefficients requires that we make an initial assumption about the form of the particular solution $y(x)$, but with the coefficients left unspecified. We then substitute the assumed expression into equation $L_n y = h$…….(1) and attempt to determine the coefficients so as to satisfy that equation. If we successful, then we have found a solution of the differential eq(1) and can use it for the particular solution $y(x)$. If we cannot determine the coefficients, then this means that there is no solution of the form that we assumed. In this case we may modify the initial assumption and try again.

We can choose the assumption of the particular solution according to the function $h(x)$ that it is given in equation and the following table illustrates how to choose:

	$h(x)$	$y(x)$
1-	$P_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$	$x^s(A_0x^n + A_1x^{n-1} + \cdots + A_n)$
2-	$P_n(x)e^{ax}$	$x^s(A_0x^n + A_1x^{n-1} + \cdots + A_n)e^{ax}$
3-	$P_n(x)e^{ax}\begin{cases} sin\beta x \\ cos\beta x \end{cases}$	$x^s[(A_0x^n + A_1x^{n-1} + \cdots + A_n)e^{ax}\ sin\beta x + (A_0x^n + A_1x^{n-1} + \cdots + A_n)e^{ax}\ cos\beta x]$

Otherwise, we use the Method of variation of constant to find the particular solution.

Notes: Here s is the smallest nonnegative integer ($s = 0, 1, 2$) that will ensure that no term in $y(x)$ is a solution of the corresponding homogeneous solution.

Example: Find the particular solution of the equation

$$y'' - 3y' + 2y = x^2 \qquad -\infty < x < \infty$$

Solution: Since $h(x) = x^2$ is the polynomial of second degree then let $y_p(x) = Ax^2 + Bx + C$ is a solution of the given equation. Since $y_p(x)$ is solution of the given equation therefore, $y_p(x)$ satisfy the equation

$\therefore\ y'_p(x) = 2Ax + B,\ \ y''_p(x) = 2A$, substitute the values of $y_p(x)$, $y'_p(x)$, and $y''_p(x)$ in the original equation we get,

$2A - 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2$ since the particular solution does not contain constant hence,

$2A - 3B + 2C = 0$

$2B - 6A = 0$

$2A = 1 \rightarrow A = \frac{1}{2},\ B = \frac{3}{2},\ C = \frac{7}{4}$

$\therefore$ the particular solution is given by $y_p(x) = \frac{1}{2}x^2 + \frac{3}{2}x + \frac{7}{4}$

Example: Find the particular solution of the equation

$$y'' - 3y' - 4y = 2\,sinx$$

Solution: Since $h(x) = 2sinx$ then let $y_p(x) = A\,sinx + Bcosx$ is a solution of the given equation. Since $y_p(x)$ is solution of the given equation therefore, $y_p(x)$ satisfy the equation

$\therefore\ \ y'_p(x) = A\,cosx - Bsinx,\quad y''_p(x) = -A\,sinx - Bcosx$, substitute the values of $y_p(x)$, $y'_p(x)$, and $y''_p(x)$ in the original equation we get,

$(-A\,sinx - Bcosx) - 3(A\,cosx - Bsinx) - 4(A\,sinx + Bcosx) = 2\,sinx$

since the particular solution does not contain constant hence,

$-3A - 5B = 0$……(1)

$3B - 5A = 2$……..(2)

Multiplying eq(1) by 3 and eq(2) by 5 and then sum two equations we get,

$-9A - 25A = 10 \rightarrow -34A = 10 \rightarrow A = -\frac{5}{17},\ B = \frac{3}{17}$

$\therefore$ the particular solution is given by $y_p(x) = -\frac{5}{17}sinx + \frac{3}{17}cosx$

Problem:

Find the particular solution of the following

1- $y'' - 3y' - 4y = 3e^{2x}$

2- $y'' - 3y' - 4y = -8e^{x}cos2x$

- ***Laplace Transform***

Linearity Property: Linearity property that the transform of a linear combination of a functions is a linear combination of the transforms. For α and β constants,

$$\frac{d}{dx}[\alpha\, f(x) + \beta\, g(x)] = \alpha\, f'(x) + \beta\, g'(x)$$

and

$$\int[\alpha\, f(x) + \beta\, g(x)]dx = \alpha \int f(x)dx + \beta \int g(x)dx$$

Provided each derivative and integral exists. In this section we will examine a special type of integral transform called the Laplace transform. In addition to have the linearity property, the Laplace transform has many other interesting properties that make it very useful in solving initial value problems.

Definition: Let f be a function defined for $t \geq 0$. Then the integral

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt \quad \text{.....(1)}$$

is said to be the Laplace transform of f, provided the integral converges.

When the defining integral (1) converges, the result is a function of s. For example, $L[f(t)] = F(s)$, $L[g(t)] = G(s)$, $L[y(t)] = Y(s)$.

Example: Evaluate $L[1]$

Solution: $L[1] = \int_0^\infty e^{-st}(1)dt = -\frac{e^{-st}}{s}\Big|_0^\infty = \frac{1}{s}$ provided $s > 0$

Example: Evaluate $L[t]$

Solution: $L[t] = \int_0^\infty e^{-st}(t)dt$ using integrating by parts

$$u = t, \quad dv = e^{-st}dt \rightarrow du = dt, \quad v = -\frac{e^{-st}}{s}$$

$$\int_0^\infty e^{-st}(t)dt = -\frac{t}{s}e^{-st}\Big|_0^\infty - \int_0^\infty \frac{-e^{-st}}{s}dt = 0 + \frac{1}{s}\int_0^\infty e^{-st}\, dt = \frac{1}{s^2},$$

provided that $s > 0$

Example: Evaluate $L[e^{-3t}]$

Solution: $L[e^{-3t}] = \int_0^\infty e^{-st}\,(e^{-3t})dt = \int_0^\infty e^{-(s+3)t}\,d = \frac{-e^{-(s+3)t}}{s+3}\Big|_0^\infty$

$= \frac{1}{s+3}$, provided $s > -3$

Example: Evaluate $L[sin2t]$

Solution: $L[sin2t] = \int_0^\infty e^{-st}\,(sin2t)dt$ using integrating by parts

$u = sin2t, \quad dv = e^{-st}dt \rightarrow du = 2\,cos2t\,dt, \quad v = -\dfrac{e^{-st}}{s}$

$\int_0^\infty e^{-st}\,(sin2t)dt = -\frac{e^{-st}}{s} sin2t\Big|_0^\infty - \int_0^\infty \frac{-2e^{-st}}{s}(cos2t)dt$

$= 0 + \frac{2}{s}\int_0^\infty e^{-st}\,(cos2t)dt$ using integrating by parts on the term $\int_0^\infty e^{-st}\,(cos2t)dt$

$u = cos2t, \quad dv = e^{-st}dt \rightarrow du = -2sin2t\,dt, \quad v = -\dfrac{e^{-st}}{s}$

$$\therefore \int_0^\infty e^{-st}\,(cos2t)dt = -\frac{e^{-st}}{s}cos2t\Big|_0^\infty - \int_0^\infty \frac{2e^{-st}}{s}(sin2t)dt$$

$= \frac{1}{s} - \frac{2}{s}L[sin2t]$ hence,

$$\int_0^\infty e^{-st}\,(sin2t)dt = \frac{2}{s^2} - \frac{4}{s^2}L[sin2t] \rightarrow [\frac{s^2+4}{s^2}]\,L[sin2t] = \frac{2}{s^2} \rightarrow$$

$L[sin2t] = \frac{2}{s^2+4}$, provided $s > 0$

Example: Evaluate $L[1+5t]$

Solution: $L[1+5t] = \int_0^\infty e^{-st}\,(1+5t)dt = \int_0^\infty e^{-st}\,dt + 5\int_0^\infty e^{-st}\,tdt$

$= \frac{1}{s} + \frac{5}{s^2}$

Example: Evaluate $L[4e^{-3t} - 10\,sin2t]$

Solution: $L[4e^{-3t} - 10\,sin2t] = \int_0^\infty e^{-st}\,(4e^{-3t} - 10\,sin2t)dt$

$= 4\int_0^\infty e^{-(s+3)t}\,dt - 10\int_0^\infty e^{-st}\,sin2t\,dt$

$= \frac{4}{s+3} - \frac{20}{s^2+4}$

Problem: Evaluate $L[t^2]$

Theorem: Transforms of some functions

a. $L[1] = \frac{1}{s}$ b. $L[t^n] = \frac{n!}{t^{n+1}}$, $n = 1, 2, 3, \ldots$ c. $L[e^{at}] = \frac{1}{s-a}$

d. $L[\ sinkt] = \frac{k}{s^2+k^2}$ e. $L[\ coskt] = \frac{s}{s^2+k^2}$ f. $L[sinh\ kt] = \frac{k}{s^2-k^2}$

g. $L[cosh\ kt] = \frac{s}{s^2-k^2}$ h. $L[\ e^{at}sinkt] = \frac{k}{(s-a)^2+k^2}$

i. $L[\ e^{at}coskt] = \frac{s-a}{(s-a)^2+k^2}$

Inverse Laplace Transform

If $F(s)$ represents the Laplace transform of a function $f(t)$ that is, $L[f(t)] = F(s)$, we then say that $f(t)$ is the inverse Laplace transform of $F(s)$ and write

$$f(t) = L^{-1}[F(s)]$$

we know that $L[1] = \frac{1}{s}$, then $1 = L^{-1}[\frac{1}{s}]$, $L[t] = \frac{1}{s^2}$ the inverse $t = L^{-1}[\frac{1}{s^2}]$, and $L[e^{-3t}] = \frac{1}{s+3}$ the inverse $e^{-3t} = L^{-1}[\frac{1}{s+3}]$.

the next theorem gives some inverse transforms

Theorem: Some inverse transforms

a. $L^{-1}\left[\frac{1}{s}\right] = 1$ b. $L^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n, n = 1, 2, 3, \ldots$ c. $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

d. $L^{-1}\left[\frac{k}{s^2+k^2}\right] = sinkt$ e. $L^{-1}\left[\frac{s}{s^2+k^2}\right] = coskt$ f. $L^{-1}\left[\frac{k}{s^2-k^2}\right] = sinh\ kt$

g. $L^{-1}\left[\frac{s}{s^2-k^2}\right] = cosh\ kt$ h. $L^{-1}\left[\frac{k}{(s-a)^2+k^2}\right] = e^{at}\ sinkt$

i. $L^{-1}\left[\frac{s-a}{(s-a)^2+k^2}\right] = e^{at}\ coskt$

Example: Evaluate $L^{-1}\left[\frac{1}{s^5}\right]$

Solution: Since $L^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$ from the above theorem $n + 1 = 5 \rightarrow n = 4$

the expression multiplying and dividing by 4!

$$\therefore\ L^{-1}\left[\frac{1}{s^5}\right] = \frac{4!}{4!}\ L^{-1}\left[\frac{1}{s^5}\right] = \frac{1}{4!}L^{-1}\left[\frac{4!}{s^5}\right] = \frac{1}{4!}\ t^4 = \frac{1}{24}\ t^4$$

Example: Evaluate $L^{-1}\left[\frac{1}{s^2+7}\right]$

Solution: Since $L^{-1}\left[\frac{k}{s^2+k^2}\right] = sinkt$ that is mean $k^2 = 7 \rightarrow k = \sqrt{7}$

the expression multiplying and dividing by $\sqrt{7}$ we get,

$$\therefore\ L^{-1}\left[\frac{1}{s^2+7}\right] = \frac{\sqrt{7}}{\sqrt{7}}\ L^{-1}\left[\frac{1}{s^2+7}\right] = \frac{1}{\sqrt{7}}\ L^{-1}\left[\frac{\sqrt{7}}{s^2+7}\right] = \frac{1}{\sqrt{7}}\ sin\sqrt{7}\ t$$

Example: Evaluate $L^{-1}\left[\frac{-2s+6}{s^2+4}\right]$

Solution: $L^{-1}\left[\frac{-2s+6}{s^2+4}\right] = L^{-1}\left[\frac{-2s}{s^2+4}\right] + L^{-1}\left[\frac{6}{s^2+4}\right] = -2L^{-1}\left[\frac{s}{s^2+4}\right] + 6L^{-1}\left[\frac{1}{s^2+4}\right]$

$$= -2L^{-1}\left[\frac{s}{s^2+4}\right] + \frac{6}{2}\ L^{-1}\left[\frac{2}{s^2+4}\right] = -2\ cos2t + 3\ sin2t$$

Example: Evaluate $L^{-1}\left[\frac{2s+3}{s^2-4s+20}\right]$

Solution: $\frac{2s+3}{s^2-4s+20} = \frac{2s+3+4-4}{s^2-4s+4+16} = \frac{2s-4+7}{s^2-4s+4+16} = \frac{2(s-2)+7}{(s-2)^2+16}$

$$= 2\left[\frac{(s-2)}{(s-2)^2+16}\right] + 7\left[\frac{1}{(s-2)^2+16}\right]$$

$$= 2\left[\frac{(s-2)}{(s-2)^2+16}\right] + \frac{7}{4}\left[\frac{4}{(s-2)^2+16}\right]$$

$$\therefore\ L^{-1}\left[\frac{2s+3}{s^2-4s+20}\right] = 2L^{-1}\left[\frac{(s-2)}{(s-2)^2+16}\right] + \frac{7}{4}L^{-1}\left[\frac{4}{(s-2)^2+16}\right]$$

$$= 2e^{2t}cos4t + \frac{7}{4}e^{2t}sin4t$$

Example: Evaluate $L^{-1}\left[\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right]$

Solution: By using the partial fractions we get

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s+4)} = \frac{A\ (s-2)(s+4)+B\ (s-1)(s+4)+C\ (s-1)(s-2)}{(s-1)(s-2)(s+4)}$$

$$= \frac{s^2(A+B+C)+s(2A+3B-3C)+(-8A-4B+2C)}{(s-1)(s-2)(s+4)}$$

$\therefore\ A + B + C = 1$ .....(1)

$2A + 3B - 3C = 6$…..(2)

$-8A - 4B + 2C = 9$…..(3)

by sum equation (2) and (3) we get

$-6A - B - C = 15$ …..(4) by sum equation (4) and (1) we get

$16 = -5A \rightarrow A = -\frac{16}{5}$ substitute the value of A in equation (1) we get

$1 + \frac{16}{5} - B = C$ substitute the value of C in equation (2) we get $B = \frac{25}{6}, C = \frac{1}{30}$

$$\therefore \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)} = \frac{-\frac{16}{5}}{(s-1)} + \frac{\frac{25}{6}}{(s-2)} + \frac{\frac{1}{30}}{(s+4)}$$

$$L^{-1}\left[\frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)}\right] = L^{-1}\left[\frac{-\frac{16}{5}}{(s-1)} + \frac{\frac{25}{6}}{(s-2)} + \frac{\frac{1}{30}}{(s+4)}\right]$$

$$= -\frac{16}{5} L^{-1}\left[\frac{1}{s-1}\right] + \frac{25}{6} L^{-1}\left[\frac{1}{s-2}\right] + \frac{1}{30} L^{-1}\left[\frac{1}{s+4}\right]$$

$$= -\frac{16}{5} e^{t} + \frac{25}{6} e^{2t} + \frac{1}{30} e^{-4t}$$

Problem:

1. Use the Laplace transform to solve the following

   a. $f(t) = 4t - 10$ b. $f(t) = t^2 + 6t - 3$ c. $f(t) = 4t^2 - 5\,sin3t$

2. Use the inverse Laplace transform to solve the following

   a. $L^{-1}[\frac{1}{s^2} - \frac{48}{s^5}]$ b. $L^{-1}[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-2}]$ c. $L^{-1}[\frac{5}{s^2+49}]$

Transforming a Derivative

In this section, we can use the Laplace transform to solve differential equations. The goal of this topic is to evaluate quantities such as $L[\frac{dy}{dt}]$, $L[\frac{d^2y}{dt^2}]$,…etc. For example, if f' is continuous for $t \geq 0$, then by using integration by parts to find

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt$$

$u = e^{-st}, \quad dv = f'(t)\, dt \ \rightarrow du = -se^{-st}dt, \ \ v = f(t)$

$L[f'(t)] = e^{-st}f(t)\Big|_0^\infty - \int_0^\infty -se^{-st}f(t)dt \ = 0 - f(0) + sL[f(t)]$

$\qquad = -f(0) + sL[f(t)] = sF(s) - f(0)$

We have to show that $L[f''(t)]$

$L[f''(t)] = \int_0^\infty e^{-st} f''(t)\, dt$ using integration by parts

$u = e^{-st}, \quad dv = f''(t)\, dt \ \rightarrow du = -se^{-st}dt, \ \ v = f'(t)$

$\int_0^\infty e^{-st} f''(t) = e^{-st} f'(t)\Big|_0^\infty - \int_0^\infty -se^{-st}f'(t)dt$

$\qquad = -f'(0) + s\int_0^\infty e^{-st}f'(t)dt \ = -f'(0) + sL[f'(t)]$

$\qquad = -f'(0) + s[sF(s) - f(0)] = s^2F(s) - sf(0) - f'(0)$

In the same away we get

$$L[f'''(t)] = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

Theorem: If $f, f', \dots, f^{n-1}$ are continuous on $[0, \infty)$ and are of exponential order and if $f^n(t)$ is piecewise continuous on $[0, \infty)$, then

$L[f^n(t)] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - \ f^{n-1}(0),$

Where $F(s) = L[f(t)]$.

The important property of the Laplace transform its very useful in solving initial value problems.

Example: Use the Laplace transform to solve the initial value problem

$$\frac{dy}{dt} - y = 1, \qquad y(0) = 0$$

Solution: We first take the Laplace transform of each member of the differential equation

$L\left[\frac{dy}{dt} - y = 1\right] = L\left[\frac{dy}{dt}\right] - L[y] = L[1]$ …..(1)

Since $L\left[\frac{dy}{dt}\right] = sY(s) - y(0) = sY(s) - 0 = sY(s),\ \ L[1] = \frac{1}{s}$

Substitute these expressions in eq(1) we get

$$sY(s) - Y(s) = \frac{1}{s} \rightarrow (s-1)Y(s) = \frac{1}{s} \rightarrow Y(s) = \frac{1}{s(s-1)}$$

$\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1} = \frac{A(s-1)+Bs}{s(s-1)} \rightarrow$

$A + B = 0$

$-A = 1 \rightarrow A = -1,\ \ B = 1$

$\therefore Y(s) = \frac{-1}{s} + \frac{1}{s-1}$ ……(2)

take the inverse Laplace transform to each member in eq(2) we get

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{-1}{s}\right] + L^{-1}\left[\frac{1}{s-1}\right]$$

$y(t) = -1 + e^{t}$

Example: Use the Laplace transform to solve

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = 0, \quad y(0) = 1,\ \ y'(0) = 0$$

Solution: $L\left[\frac{d^2y}{dt^2}\right] + 5L\left[\frac{dy}{dt}\right] + 4L[y] = L[0]$

Since $L\left[\frac{d^2y}{dt^2}\right] = s^2Y(s) - sy(0) - y'(0),\ \ \ L\left[\frac{dy}{dt}\right] = sY(s) - y(0)$

$\therefore s^2Y(s) - sy(0) - y'(0) + 5(sY(s) - y(0)) + 4Y(s) = 0 \rightarrow$

$$Y(s) = \frac{s+5}{s^2+5s+4} = \frac{s+5}{(s+4)(s+1)}$$

$$\frac{s+5}{(s+4)(s+1)} = \frac{A}{s+4} + \frac{B}{s+1} = \frac{A(s+1)+B(s+4)}{(s+4)(s+1)}$$

$A + B = 1$…..(1)

$A + 4B = 5$…...(2)

Multiply eq(2) by -1 and sum with eq(2) we get

$$B = \frac{4}{3}, \quad A = \frac{-1}{3}$$

$$\therefore Y(s) = \frac{-1/3}{s+4} + \frac{4/3}{s+1}$$

take the inverse Laplace transform to both side we get

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{-1/3}{s+4}\right] + L^{-1}\left[\frac{4/3}{s+1}\right] \rightarrow y(t) = -\frac{1}{3}e^{-4t} + \frac{4}{3}e^{-t}$$

Example: Use the Laplace transform to solve the initial value problem

$$\frac{dy}{dt} + 3y = 13sin2t, \qquad y(0) = 6$$

Solution: Take the Laplace transform to both sides of equation we get

$$L[\frac{dy}{dt}] + 3L[y] = 13L[sin2t]$$

Since $L\left[\frac{dy}{dt}\right] = sY(s) - y(0) = sY(s) - 6$

$$sY(s) - 6 + 3Y(s) = \frac{26}{s^2+4} \rightarrow Y(s) = \frac{6}{s+3} + \frac{26}{(s+3)(s^2+4)} \rightarrow$$

$$Y(s) = \frac{6s^2+50}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4} \rightarrow$$

$$Y(s) = \frac{(A+B)s^2 + (3s+C)s + 4A + 3C}{(s+3)(s^2+4)}$$

$A + B = 6$…..(1) $\rightarrow A = 6 - B$

$3B + C = 0$…..(2)

$4A + 3C = 50$…..(3) $\rightarrow A = \frac{50-3C}{4} \rightarrow \frac{50-3C}{4} = 6 - B \rightarrow 3C - 4B = 26$

$3B + C = 0 \rightarrow C = -3B$

$\therefore 3(-3B) - 4B = 26 \rightarrow B = -2, \quad C = 6, \qquad A = 8$

$$Y(s) = \frac{6s^2 + 50}{(s+3)(s^2+4)} = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}$$

take the inverse Laplace transform we get

$y(t) = 8e^{-3t} - 2cos2t + 3sin2t$

Example: Use the Laplace transform to solve the initial value problem

$$y''' + 3y' = 9t^2 - 12t + 6, \quad y''(0) = -4, \; y'(0) = 0, \; y(0) = 3$$

Solution: Take the Laplace transform to both sides of equation we get

$L[y'''] + 3L[y'] = 9L[t^2] - 12L[t] + L[6]$

Since $L[y'''] = s^3Y(s) - s^2y(0) - sy'(0) - y(0), \; L[y'] = sY(s) - y(0)$

$$(s^3 + 3s)Y(s) = 3s^2 + 5 + \frac{18}{s^3} - \frac{12}{s^2} + \frac{6}{s} = \frac{3s^5 + 5s^3 + 18 - 12s + 6s^2}{s^3}$$

$$(s^3 + 3s)Y(s) = \frac{3s^5 - 4s^3 + 18 - 12s + 6s^2 + 9s^3}{s^3}$$

$$(s^3 + 3s)Y(s) = \frac{s^2(3s^3 - 4s + 6) + 3(s^3 - 4s + 6)}{s^3}$$

$$Y(s) = \frac{(s^2+3)(3s^3 - 4s + 6)}{(s^2+3)s^4} = \frac{3}{s} - \frac{4}{s^3} + \frac{6}{s^4}$$

take the inverse Laplace transform on both sides we get

$y(t) = 3 - 2t^2 + t^3$

Problem: Use the Laplace transform to solve the following

1- $y'' + 9y = e^t, \quad y(0) = 0,\ y'(0) = 0$

2- $y'' + y = \sqrt{2}\, sin\sqrt{2}\, t,\ \ y(0) = 10,\ y'(0) = 0$

Solution of some problems

1- Find the general solution of the following equation

$$\frac{d^2y}{dx^2} + 4y = sec2x$$

Solution

The characteristic equation of the homogeneous equation $\frac{d^2y}{dx^2} + 4y = 0$ is given by the form $P_2(\lambda) = \lambda^2 + 4 = 0$ and the roots are $\lambda = \mp 2i$ therefore, the general solution for the homogeneous equation is $y_c(x) = c_1 sin2x + c_2 cos2x$, since $w(g_1, g_2)(x) = \begin{vmatrix} sin2x & cos2x \\ 2cos2x & -2sin2x \end{vmatrix} = -2$ substitute the values of $w(g_1, g_2) = -2,\ g_1 = sin2x,\ g_2 = cos2x,\ h(x) = sec2x$ in equation (*) we get the particular of the solution of the given non homogeneous equation

$$y_p(x) = \int_{x_o}^{x} \frac{(sin2t\ cos2x - sin2x\ cos2t)}{-2}\ sec2t\ dt = -\frac{1}{2}\left[\int_0^x \left(cos2x\ \frac{sin2t}{cos2t}\right) dt - \int_0^x sin2x\ \frac{cos2t}{cos2t} dt\right]$$

$$= -\frac{1}{2}[cos2x\left(-\frac{1}{2}ln|cos2t|\right)\Big|_0^x - sin2x \int_0^x dt]$$

$$= -\frac{1}{2}[cos2x\left(-\frac{1}{2}(ln|cos2x| - ln|1|)\right) - sin2x(t)\Big|_0^x$$

$$= \frac{1}{4}cos2x\ ln|cos2x| - xsin2x$$

The general solution of the given equation is

$$y(x) = c_1 sin2x + c_2 cos2x + \frac{1}{4}cos2x\ ln|cos2x| - xsin2x$$

2- Find the particular solution of the following equation

$$y''' - 4y' = sinx$$

Solution

Since $h(x) = sinx$ then let $y_p(x) = Asinx + Bcosx$ is a solution of the given equation. Since $y_p(x)$ is solution of the given equation therefore, $y_p(x)$ satisfy the equation

$\therefore y_p'(x) = A\ cosx - B\ sinx, y_p''(x) = -A\ sinx - B\ cosx,$

$y_p'''(x) = -A\,cosx + B\,sinx,$ substitute the values of $y_p'(x),$ and $y_p'''(x)$ in the original equation we get,

$-A\,cosx + B\,sinx - 4(A\,cosx - B\,sinx) = sinx$ since the particular solution does not contain constants hence,

$-5A = 0 \rightarrow A = 0$

$5B = 1 \rightarrow B = \frac{1}{5}$

$\therefore$ the particular solution is given by $y_p(x) = \frac{1}{5}cosx$

3- Find the particular solution of the following equation

$y'' - 3y' - 4y = 3e^{2x}$

Solution

Since $h(x) = 3e^{2x}$ then let $y_p(x) = Ae^{2x}$ is a solution of the given equation. Since $y_p(x)$ is solution of the given equation therefore, $y_p(x)$ satisfy the equation

$\therefore y_p'(x) = 2Ae^{2x}\,, y_p''(x) = 4A\,e^{2x},$

substitute the values of $y_p'(x),$ and $y_p''(x)$ in the original equation we get,

$4A\,e^{2x} - 3(2Ae^{2x}) - 4\,Ae^{2x} = 3e^{2x}$ since the particular solution does not contain constants hence,

$-6A = 3 \rightarrow A = -\frac{1}{2}$

$\therefore$ the particular solution is given by $y_p(x) = -\frac{1}{2}e^{2x}$

4- Find the particular solution of the following equation

$y'' - 3y' - 4y = -8e^{x}cos2x$

Solution